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Geometry of the Aharonov–Bohm Effect

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Abstract We show that the connection responsible for any Abelian or non-Abelian Aharonov–Bohm effect with n parallel "magnetic" flux lines in $\mathbb{R}^3$, lies in a trivial G-principal bundle $P \to M$, i.e. P is isomorphic to the product $M \times G$, where G is any path connected topological group; in particular a connected Lie group. We also show that two other bundles are involved: the universal covering space $\tilde{M} \to M$, where path integrals are computed, and the associated bundle $P \times_G \mathbb{C}^m \to M$, where the wave function and its covariant derivative are sections.

Keywords Aharonov–Bohm effect · Fibre bundle theory · Gauge invariance

As it is well known, the magnetic Aharonov–Bohm (A-B) effect [1, 2] is a gauge invariant, non local quantum phenomenon, with gauge group $U(1)$, which takes place in a non-simply connected space. It involves a magnetic field in a region where an electrically charged particle obeying the Schroedinger equation cannot enter, i.e. the ordinary 3-dimensional space minus the space occupied by the solenoid producing the field; in the ideal mathematical limit, the solenoid is replaced by a flux line. Locally, the particle couples to the magnetic potential $\vec{A}$ but not to the magnetic field $\vec{B}$; however, the effect is gauge invariant since it only depends on the flux of $\vec{B}$ inside the solenoid.

The fibre bundle theoretic description of this kind of phenomena has proved to be very useful to obtain a more profound insight of the relation between physical processes and

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pure mathematics [3]. In the present case, since by symmetry, the dimension along the flux line can be ignored, the problem reduces to the effect on the charged particle of an Abelian connection in a $U(1)$-bundle with base space the plane minus a point. The wave function representing the particle is a section of an associated vector bundle. As shown in Ref. [4], the bundle turns out to be trivial and then its total space is isomorphic to the product $(\mathbb{R}^2 - \{\text{point}\}) \times U(1)$ (see also Ref. [5]). Since $\mathbb{R}^2$ is topologically equivalent to an open disk, and $U(1)$ is the unit circle, the bundle structure is summarized by

$$U(1) \to T^2_{\circ *} \to D^2_{\circ *} \tag{1}$$

where $D^2_{\circ *}$ is the open disk minus a point and $T^2_{\circ *}$ is the open solid 2-torus minus a circle.

As suggested by Wu and Yang [6], Yang–Mills fields can give rise to non-Abelian A-B effects. In Ref. [6] the authors studied an $SU(2)$ gauge configuration leading to an A-B effect; later, several authors studied the effect with gauge groups $SU(3)$ [7] and $U(N)$ [8]. Also, in Refs. [9–14] the effect was studied in the context of gravitation theory. In all these examples, as in the magnetic case, there is a principal bundle structure

$$\xi : G \to P \xrightarrow{\pi_G} M \tag{2}$$

whose total space P, where the connection giving rise to the effect lies, is however never specified. In this note we prove the following:

Theorem 1 *Let G be a path connected topological group (for example a connected Lie group) and ω the connection responsible for an A-B effect, with n ($n = 1, 2, 3, \ldots$) parallel flux lines in $\mathbb{R}^3$. Then the bundle ξ is trivial, i.e. isomorphic to the product bundle.*

Proof The classification of bundles over $\mathbb{R}^3\backslash\{n \text{ parallel lines}\}$ is the same as that over $\mathbb{R}^2\backslash\{n \text{ points}\}$ which is topologically equivalent to $D^2_\circ\backslash\{n \text{ points}\} \equiv D^2_{\circ * n}$. Denote this set of points by $\{b_1, \ldots, b_n\}$. By symmetry along the dimension of the flux tubes, $D^2_{\circ * n}$ is the space where it can be considered that the charged particles move.

Let x_0 be a point in $D^2_{\circ * n}$. We construct a bouquet of n loops $\gamma_1, \gamma_2, \ldots, \gamma_n$ through x_0, with the k-th loop surrounding the point b_k, $k = 1, 2, \ldots, n$. This space is homeomorphic to the wedge product (or reduced join) $S^1 \vee \ldots \vee S^1 \equiv \vee_n S^1 \equiv S^1_{(1)} \vee \ldots \vee S^1_{(n)}$ of n circles [15], and the classification of bundles over $D^2_{\circ * n}$ is the same as that over $\vee_n S^1$, namely

$$\mathcal{B}_{D^2_{\circ * n}}(G) = \mathcal{B}_{\vee_n S^1}(G) \tag{3}$$

where $\mathcal{B}_M(G)$ is the set of isomorphism classes of G-bundles over M [16].

By explicit construction we shall prove that, up to isomorphism, the unique G-bundle over $\vee_n S^1$ is the product bundle $\vee_n S^1 \times G$ (which is a purely topological result). With this aim, we cover the circle $S^1_{(k)}$ with two open sets U_{k+} and U_{k-} such that

$$U_{k+} \cap U_{k-} \simeq \{x_0, a_k\}, \quad k = 1, \ldots, n, \tag{4}$$

$$U_{i+} \cap U_{j+} \simeq U_{i-} \cap U_{j-} \simeq U_{i+} \cap U_{j-} \simeq \{x_0\}, \quad i, j = 1, \ldots, n, \ i \neq j \tag{5}$$

where $\simeq$ denotes homotopy equivalence, and $a_k \in S^1_{(k)}$ with $a_k \neq x_0$. We then have

$$\binom{2n}{2} = \frac{(2n)!}{(2n-2)!2!} = 2n^2 - n \tag{6}$$

transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G \tag{7}$$

which up to homotopy are given by

$$g_{kk} : \{x_0, a_k\} \to G, \qquad x_0 \mapsto g_0, \qquad a_k \mapsto g_k, \quad k = 1, \dots, n \tag{8}$$

and

$$\begin{aligned} &g_{i+,j+}, g_{i-,j-}, g_{i+,j-} : \{x_0\} \to G, \quad i, j = 1, \dots, n,\ i \neq j, \\ &\qquad x_0 \mapsto g_{ij++}, \qquad x_0 \mapsto g_{ij--}, \qquad x_0 \mapsto g_{ij+-}. \end{aligned} \tag{9}$$

Let $g'_{kk} : \{x_0, a_k\} \to G$ be given by $g'_{kk}(x_0) = g'_0$, $g'_{kk}(a_k) = g'_k$. Since G is path connected, there exist continuous paths

$$c_0^{(k)} : [0, 1] \to G, \qquad t \mapsto c_0^{(k)}(t) \quad \text{with } c_0^{(k)}(0) = g_0,\ c_0^{(k)}(1) = g'_0$$

and

$$c_k^{(k)} : [0, 1] \to G, \qquad t \mapsto c_k^{(k)}(t) \quad \text{with } c_k^{(k)}(0) = g_k,\ c_k^{(k)}(1) = g'_k.$$

Then the continuous function

$$H_k : \{x_0, a_k\} \times [0, 1] \to G$$

given by

$$H_k(x_0, t) = c_0^{(k)}(t), \qquad H_k(a_k, t) = c_k^{(k)}(t) \tag{10}$$

is a homotopy between g_{kk} and g'_{kk} since

$$H_k(x_0, 0) = g_0 \quad \text{and} \quad H_k(a_k, 0) = g_k$$

i.e.

$$H_k|_{\{x_0,a_k\}\times\{0\}} = g_{kk}$$

and

$$H_k(x_0, 1) = g'_0 \quad \text{and} \quad H_k(a_k, 1) = g'_k$$

i.e.

$$H_k|_{\{x_0,a_k\}\times\{1\}} = g'_{kk}.$$

Then the homotopy class of maps of g_{kk} has only one element, namely

$$[g_{kk}]_\sim = \{g_{kk}\}. \tag{11}$$

Similarly, let $g_{i\alpha,j\beta}$ with $\alpha, \beta \in \{+, -\}$ be any of the functions in (9), and let $g'_{i\alpha,j\beta} : \{x_0\} \to G$ be given by

$$g'_{i\alpha,j\beta}(x_0) = g'_{ij,\alpha\beta}; \tag{12}$$

then the map

$$H_{i\alpha,j\beta}:\{x_0\}\times[0,1]\to G,\qquad H_{i\alpha,j\beta}(x_0,t)=c_{i\alpha,j\beta}(t) \tag{13}$$

with $c_{i\alpha,j\beta}:[0,1]\to G$ a continuous path in G satisfying $c_{i\alpha,j\beta}(0)=g_{ij,\alpha,\beta}$ and $c_{i\alpha,j\beta}(1)=g'_{ij,\alpha\beta}$, is a homotopy between $g_{i\alpha,j\beta}$ and $g'_{i\alpha,j\beta}$ i.e.

$$[g_{i\alpha,j\beta}]_{\sim}=\{g_{i\alpha,j\beta}\}. \tag{14}$$

Since homotopy equivalent transition functions give rise to isomorphic bundles, and the product bundle always exists, we have the desired result:

$$\mathcal{B}_{\vee_n S^1}(G)=\{[G\to\vee_n S^1\times G\to\vee_n S^1]\} \tag{15}$$

where [ ] denotes here the equivalence class of bundles isomorphic to the product bundle. □

For the cases of the examples in Refs. [1, 6, 7], we have, respectively,

$$\mathcal{B}_{D^2_{\circ *}}(U(1))=\{[U(1)\to D^2_{\circ *}\times U(1)\to D^2_{\circ *}]\}, \tag{16}$$

$$\mathcal{B}_{D^2_{\circ *}}(SU(2))=\{[SU(2)\to D^2_{\circ *}\times SU(2)\to D^2_{\circ *}]\}, \tag{17}$$

and

$$\mathcal{B}_{D^2_{\circ *2}}(SU(3))=\{[SU(3)\to D^2_{\circ *2}\times SU(3)\to D^2_{\circ *2}]\}. \tag{18}$$

In the gravitational case, the gauge group is $SL(2,\mathbb{C})$ [17], which is the universal covering group of the connected component of the Lorentz group $L^{\uparrow}_{+}$; then, for weak gravitational fields with a distribution of n gravitomagnetic flux lines as above we would have the trivial bundle

$$SL(2,\mathbb{C})\to D^2_{\circ *n}\times SL(2,\mathbb{C})\to D^2_{\circ *n}. \tag{19}$$

We want to stress that, even if the A-B connection is flat (though not exact), one of the sufficient conditions for the automatic triviality of the A-B bundle fails: though paracompact, the bouquet $\vee_n S^1$ is not simply connected. (See Corollary 9.2. in Ref. [20, p. 92].) In addition, there is not a priori any physical reason why, under the specified conditions on the "magnetic flux", the corresponding A-B bundle should be trivial.

It is interesting to notice that there are two other fibre bundles related to the A-B effect. The first bundle is the *universal covering space* [18] of the base manifold ("laboratory" or physical space where the particles coupled to the A-B potential move) which is the $\pi_1(M;x_0)$-(non-trivial) bundle $\xi_c:\tilde{M}\xrightarrow{\pi}M$, where $\pi_1(M;x_0)$ ($\equiv\pi_1(M)$ if M is connected) is the fundamental group of M. In our case,

$$\pi_1(\vee_n S^1;x_0)\cong\pi_1(\mathbb{R}^2\backslash\{n\text{ points}\};x_0)\cong\langle\{c_1,\ldots,c_n\}\rangle$$

is the freely generated group with n non-commuting generators $c_1,\ldots,c_n$. In particular, for the original Abelian A-B effect with group $U(1)$, $\tilde{\mathbb{R}}^{2*}=RS(\mathrm{Log})$: the Riemann surface of the logarithm, and $\pi_1(\mathbb{R}^{2*})\cong\mathbb{Z}$.

The *particle propagator* in M, $K(x'',t'';x',t')$ with $t''>t'$, is a sum of homotopy propagators [7] multiplied by corresponding gauge factors [19]: the former are given by unrestricted path integrals computed in $\tilde{M}$, the paths in these path integrals project onto the corresponding homotopy classes of paths in the non-simply connected space M; the latter are Wilson loops given by

$$T \exp \int_{\pi(c)} \vec{A} \cdot d\vec{l}$$

where T denotes time order, $\vec{A}$ is the A-B potential, and c is a loop in $\tilde{M}$ beginning and ending respectively at y_0 and y'' in $\pi^{-1}(\{x''\}) \overset{\Psi}{\cong} \pi_1(M)$, with y_0 fixed and arbitrary. Then one has the group homomorphism (many-to-one or one-to-one)

$$y'' \xrightarrow{\Psi} \Psi(y'') \xrightarrow{\varphi} T \exp \int_{\pi(c)} \vec{A} \cdot d\vec{l} \in G \tag{20}$$

whose image in G, $\varphi(\pi_1(M))$, responsible for the A-B effect, is the *holonomy* of the connection [20].

The second bundle is the *associated vector bundle* $\xi_{\mathbb{C}^m} : \mathbb{C}^m - P_{\mathbb{C}^m} \xrightarrow{\pi_{\mathbb{C}^m}} M$ (for spinless particles $m = 1$) where

$$\begin{aligned} P_{\mathbb{C}^m} &= (M \times G) \times_G \mathbb{C}^m \\ &= \{[((x,g),\vec{z})]\}_{((x,g),\vec{z}) \in (M\times G)\times_G \mathbb{C}^m}, [((x,g),\vec{z})] = \{((x,gg'), g'^{-1}\vec{z})\}_{g' \in G}. \end{aligned} \tag{21}$$

$\xi_{\mathbb{C}^m}$ is trivial since ξ is trivial, and the quantum mechanical *wave functions* of the particles are global sections of $\xi_{\mathbb{C}^m}$:

$$\psi \in \Gamma(\xi_{\mathbb{C}^m})$$

i.e. $\psi : M \to P_{\mathbb{C}^m}$ with $\pi_{\mathbb{C}^m} \circ \psi = Id_M$.

Notice that while the propagator is computed in ξ_c, the wave function lies in $\xi_{\mathbb{C}^m}$, with

$$\psi(x'', t'') = \int_M dx' K(x'', t''; x', t') \psi(x', t'). \tag{22}$$

If ω is the A-B connection in P, then the coupling $\omega - \psi$ is the covariant derivative

$$\nabla^{\omega}_{V} \psi = \psi_{V^{\uparrow}(\gamma_\psi)} \in \Gamma(\xi_{\mathbb{C}^m}), \tag{23}$$

where V is a vector field in M, $V^{\uparrow}$ its horizontal lifting in P by ω, γ_ψ and $V^{\uparrow}(\gamma_\psi)$ are equivariant functions from P to $\mathbb{C}^m$ with $\gamma_\psi(p) = \vec{z}$ where $\psi(\pi_G(p)) = [p, \vec{z}]$, and $\psi_{V^{\uparrow}(\gamma_\psi)}(x) = [p, V^{\uparrow}(\gamma_\psi)(p)]$ for any $p \in \pi_G^{-1}(\{x\})$. Locally, of course, $\nabla^{\omega}_{V}\psi$ reproduces the usual minimal coupling between $\vec{A}$ and ψ.

Finally, though $\varphi : \pi_1(M) \to G$ is a group homomorphism, and $\tilde{M} \xrightarrow{f} M \times G$ given by $f(y) = (\pi(y), 1)$ is a canonical map, there is no bundle map between ξ_c and ξ: the pair of functions $(f \times \varphi, f)$ is *not* a principal bundle homomorphism.

In summary, the three bundles are related by the following diagram:

$$\begin{array}{ccccc} \pi_1(M) & \xrightarrow{\varphi} & G & & \mathbb{C}^m \\ \downarrow & & \downarrow & & | \\ \tilde{M} & \xrightarrow{f} & M \times G & \xrightarrow{\iota} & (M \times G) \times_G \mathbb{C}^m \\ \downarrow \pi & & \downarrow \pi_G & & \downarrow \pi_{\mathbb{C}^m} \uparrow \psi \uparrow \nabla^{\omega}_{V}\psi \\ M & = & M & = & M \end{array}$$

where ι is the canonical injection of the bundle ξ into its associated bundle i.e. $\iota(p) = [p, 0]$.

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